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Variational Theory and Differential Operators in the Framework of Special Kawaguchi Spaces

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Abstract:

This study is devoted to the advancement of the theory of variations, incorporating the foundational concepts and precise formulations of covariant, Lie, and apparent differentials. Throughout the paper, several significant theorems are rigorously established, providing a structured approach to the understanding and application of these differentials in the broader context of differential geometry. Particular emphasis is placed in section 3 on the investigation of variational phenomena within the setting of special Kawaguchi spaces, highlighting unique geometric properties and extending classical results. The framework developed herein offers new insights and tools for the study of generalized spaces, potentially impacting future research in both pure and applied mathematics.

1. Introduction:

1.1 Overview:

The theory of variations occupies a central position in the development of modern differential geometry and its applications to mathematical physics. By studying how small changes in functions and geometrical structures affect given functionals, variational theory provides profound insights into the intrinsic nature of geometric spaces. The present research focuses on the study of variations through the lens of covariant, Lie, and apparent differentials, establishing rigorous connections between these differential operators and broader geometric constructs. Special emphasis is placed on the framework of special Kawaguchi spaces, which serve as a natural generalization of classical spaces, offering a rich field for the application and extension of variational methods.

1.2 Historical Background:

The origins of variational theory can be traced back to the works of Euler, Lagrange, and Hamilton, who pioneered the calculus of variations in the 18th and 19th centuries. Their foundational contributions established the framework for understanding critical points of functionals, leading to the formulation of Euler-Lagrange equations, which underpin much of classical mechanics and field theory. In the 20th century, the evolution of differential geometry, particularly through the development of concepts such as covariant derivatives, Lie derivatives, and the notion of fiber bundles, significantly enriched the mathematical toolkit available for variational studies. Kawaguchi spaces, introduced as a generalization of Finsler spaces, provided an even more flexible environment to model variational principles in a geometrically coherent manner. This progression has opened new avenues for theoretical exploration and practical applications, especially in fields involving generalized metrics and higher-order structures.

1.3 Recent Developments and Applications:

In recent decades, the theory of variations has found renewed relevance across diverse scientific domains. In theoretical physics, variational principles continue to serve as the foundational mechanism for formulating modern field theories, including general relativity and gauge theories. Advanced studies in elasticity, fluid mechanics, and material science utilize variational techniques to model complex systems characterized by non-linear behaviors and anisotropic structures. Simultaneously, differential operators such as the Lie and covariant differentials have gained prominence in contemporary research areas like control theory, robotics, and computational geometry, where the understanding of infinitesimal transformations is crucial. The exploration of special Kawaguchi spaces has particularly advanced the modeling of complex systems with variable structural properties, offering novel solutions to problems where classical Riemannian or Finslerian frameworks prove insufficient. The study of apparent differentials, in particular, has found utility in analyzing dynamic systems where traditional differential measures are inadequate.

1.4 Research Objectives:

The primary objective of this research is to systematically develop the theory of variations in the context of special Kawaguchi spaces by employing covariant, Lie, and apparent differentials. Specifically, this study aims to:

•Formally define and analyze the properties of these differentials within generalized geometric structures.

•Establish key theorems that provide a foundational understanding of variational behavior in special Kawaguchi spaces.

•Investigate the implications of these findings in broader geometric contexts and identify potential applications in modern mathematical and physical theories.

Through a rigorous and comprehensive approach, the present work aspires to contribute meaningfully to the ongoing development of differential geometry and its variational aspects.

1.5 Preliminary Concepts on Covariant Differentials:

In order to develop the variational theory in special Kawaguchi spaces, we introduce a class of covariant differentials constructed using a skew-symmetric tensor defined by:

(1.1) $$H_{ab} = \frac{1}{2}(\dot{\partial}_a A_b - \dot{\partial}_b A_a).$$

We define a new covariant differential and corresponding derivatives as follows:

(1.2) $$Dv^a = dv^a + \Gamma^{*a}_{bc}\, v^b dx^c + C^a_{bc}\, v^b \delta\dot{x}^c$$

and

(1.3) $$Dv^a = dx^b \dot{\nabla}_b v^a + \delta\dot{x}^c \widehat{\nabla}_b v^a$$

where the operations $\dot{\nabla}_b$ and $\widehat{\nabla}_b$ are considered with respect to x^b and $\dot{x}^b$, respectively.

The operations $\dot{\nabla}_b$ and $\widehat{\nabla}_b$ are subsequently introduced and defined as:

(1.4) $$\dot{\nabla}_b v^a_c = \partial_b v^a_c - \partial_d v^a_c\, \dot{\partial}_b \Gamma^d + \Gamma^{*a}_{db}\, v^d_c - \Gamma^{*d}_{cb}\, v^a_d$$

and

(1.5) $$\widehat{\nabla}_b v^a_c = \dot{\partial}_b v^a_c + C^a_{db} v^d_c - C^d_{cb} v^a_d,$$

where Γ^d and Γ^{*a}_{db} represent connection parameters, C^a_{db} is a tensor, and $\delta\dot{x}^a$ is given by:

(1.6) $$\delta\dot{x}^a = \delta x^a + (\dot{\partial}_b \Gamma^a) dx^b.$$

The commutative formula can be expressed in terms of the curvature tensors K^a_{bc} and R^{*a}_{bcd} as:

(1.7) $$\dot{\nabla}_{[b}\dot{\nabla}_{c]} v^a_d = \frac{1}{2}(R^{*a}_{bce}\, v^e_d - R^{*e}_{bcd}\, v^a_e + K^e_{bc}\, \dot{\partial}_e v^a_d) + S^{*e}_{bc}\, \dot{\nabla}_e v^a_d,$$

Where

(1.8) $$R^{*a}_{bcd} = 2(\partial_{[b} \Gamma^{*a}_{|d|c} - \dot{\partial}_e \Gamma^{*a}_{d[c}\, \dot{\partial}_{b]} \Gamma^e - \Gamma^{*e}_{d[c} \Gamma^{*a}_{|e|b]}).$$

In particular, the quantities K^a_{bc} and Γ^a are related as:

(1.9) $K^a_{bc} = 2\partial_{[c}\dot{\partial}_{b]}\Gamma^a + \dot{\partial}_e\dot{\partial}_{[c}\Gamma^a\dot{\partial}_{b]}\Gamma^e)$

The quantity S^{*a}_{bc} is also defined in terms associated connection parameters:

(1.10) $S^{*a}_{bc} = \frac{1}{2}(\Gamma^{*a}_{bc} - \Gamma^{*a}_{cb})$.

2. Basic Definitions of Apparent Differentials:

In this section, we introduce the concept of apparent differentials within the framework of special Kawaguchi geometry. These differentials provide a structural tool to analyze variations and geometric properties that are not evident through standard covariant or Lie differentials. The formal definitions and foundational results will be developed to support the variational theory established in Section 1.5.

Let Ω be a geometric object defined in the neighborhood of a point $P \equiv (x^a)$, and consider an infinitesimal transformation of the form

(2.1) $x^{*a} = x^a + v^a(x)d\tau$,

where $d\tau$ is an infinitesimal constant and $v^a(x)$ is a vector field defined over the region under consideration.

Differentiating equation (2.1), we obtain

(2.2) $\dot{x}^{*a} = \dot{x}^a + (\partial_b v^a)\,\dot{x}^b d\tau$,

By virtue of equations (2.1) and (2.2), it follows that

(2.3) $dx^a = v^a(x)d\tau$,

and

(2.4) $d\dot{x}^a = (\partial_b v^a)\dot{x}^b d\tau$.

Now, let $X^a(x,\dot{x})$ be a contravariant vector field that is homogeneous of degree zero in $\dot{x}^a$. Then, to first order in $d\tau$, the value of the vector field $X^a(x^*,\dot{x}^*)$ at the new element is given by

(2.5) $X^a(x^*,\dot{x}^*) = X^a(x^c + v^c d\tau, \dot{x}^c + (\partial_b v^c)\dot{x}^b d\tau)$,

i.e.

(2.6) $X^a(x^*,\dot{x}^*) = X^a(x,\dot{x}) + \partial_c X^a(x,\dot{x})v^c d\tau + \dot{\partial}_c X^a(x,\dot{x})(\partial_b v^c)\dot{x}^b d\tau$.

Thus, the difference between $X^a(x^*,\dot{x}^*)$ and $X^a(x,\dot{x})$ is

(2.7) $\delta_1 X^a = \partial_c X^a v^c d\tau + \dot{\partial}_c X^a(\partial_b v^c)\dot{x}^b d\tau$.

Similarly, for a covariant vector field $X_a(x,\dot{x})$, we have

(2.8) $\delta_1 X_a = \partial_c X_a v^c d\tau + \dot{\partial}_c X_a(\partial_b v^c)\dot{x}^b d\tau$.

Consequently, equation (2.1) yields

(2.9) $\partial_b x^{*a} = \delta^a_b + (\partial_b v^a)d\tau$

and

(2.10) $\partial^*_b x^a = \delta^a_b - (\partial_b v^a)d\tau$.

Remark 2.1: The partial derivative with respect to x^{*a} is denoted by ∂^*_a.

Next, if we take $\delta_2 X^a$ as the difference of $X^{*a}(x^*,\dot{x}^*) - X^a(x,\dot{x})$, then $\delta_2 X^a$ reduces in the form

(2.11) $\delta_2 X^a = (\partial_b v^a)X^b d\tau$.

Similarly, for a covariant vector field $X_a(x,\dot{x})$, we have

(2.12) $\delta_2 X_a = -(\partial_a v^b)X_b d\tau$.

Let us now denote by $X^a(x,\dot{x}||x^*,\dot{x}^*)$ the vector transported parallelly from $(x,\dot{x})$ to $(x^*,\dot{x}^*)$. Then, from equation (1.2), we obtain

(2.13) $DX^a = dX^a + \Gamma^{*a}_{bc}\,X^b dx^c + C^a_{bc}\,X^b\delta\dot{x}^c = 0$.

Further, if we take $\delta_3 X^a$ as the difference of $X^a(x,\dot{x}||x^*,\dot{x}^*) - X^a(x,\dot{x})$, then $\delta_3 X^a$ takes the form

(2.14) $\delta_3 X^a = -(\Gamma^{*a}_{bc}\,X^b dx^c + C^a_{bc}\,X^b\delta\dot{x}^c)$.

As a consequence of equations (1.6), (2.3) and (2.4), we have

(2.15) $\delta\dot{x}^a = (\nabla_b v^a)\dot{x}^b d\tau$,

where $\nabla_b v^a$ is called a Kawaguchi covariant derivative and it is defined as

(2.16) $\nabla_b v^a = \partial_b v^a + (\dot{\partial}_b \dot{\partial}_c \Gamma^a) v^c$.

Inserting equations (2.3) and (2.15) into the equation (2.14), we have

(2.17) $\delta_3 X^a = -[\Gamma^{*a}_{bc} X^b v^a d\tau + C^a_{bc} X^b (\nabla_d v^c)\dot{x}^d d\tau]$.

Similarly, for a covariant vector $X_a(x, \dot{x})$, we have

(2.18) $\delta_3 X_a = -[\Gamma^{*b}_{ac} X_b v^c d\tau + C^b_{ac} X_b (\nabla_d v^c)\dot{x}^d d\tau]$.

Definition 2.1:

Let Ω be a geometric object defined in the neighborhood of a point $P \equiv (x^a)$. At the point $Q \equiv (x^a + v^a d\tau)$, the value of the field at rest is denoted by $\Omega + \delta_1\Omega$.

Definition 2.2:

Let Ω be a geometric object defined in the neighborhood of a point $P \equiv (x^a)$. At the point $Q \equiv (x^a + v^a d\tau)$, the change in the field resulting from dragging along the vector field $v^a d\tau$ is denoted by $\Omega + \delta_2\Omega$.

Definition 2.3:

Let Ω be a geometric object defined near a point $P \equiv (x^a)$. At the point $Q \equiv (x^a + v^a d\tau)$, the change resulting from a parallel displacement of the field along $v^a d\tau$ is denoted by $\Omega + \delta_3\Omega$.

Definition 2.4:

For a geometric object Ω defined near $P \equiv (x^a)$, the covariant differential D is defined as

(2.19) $D\Omega = \delta_1\Omega - \delta_3\Omega$.

Definition 2.5:

For a geometric object Ω, the Lie differential is defined as

(2.20) $L_v\Omega d\tau = \delta_1\Omega - \delta_2\Omega$.

Definition 2.6:

The apparent differential D_s is defined as

(2.21) $D_s\Omega d\tau = \delta_3\Omega - \delta_2\Omega$.

Hence, we have the following theorems:

Theorem 2.1:

In a special Kawaguchi space, if $D\Omega = 0$, then the relation $L_v\Omega d\tau = D_s\Omega d\tau$ holds.

Proof:

Setting $D\Omega = 0$ in equation (2.19) and using equations (2.20) and (2.21) yields

(2.24) $L_v\Omega d\tau = D_s\Omega d\tau$.

Thus, the theorem is established.

Theorem 2.2:

In a special Kawaguchi space, if $L_v\Omega d\tau = 0$, then the relation $D\Omega + D_s\Omega d\tau = 0$ holds.

Proof:

Substituting $L_v\Omega d\tau = 0$ into equation (2.20), we obtain

(2.25) $\delta_1\Omega = \delta_2\Omega$,

Inserting this into (2.19), we get

(2.26) $D\Omega + (\delta_3\Omega - \delta_2\Omega) = 0$,

Thus,

(2.27) $D\Omega = D_s\Omega d\tau$

as required.

Theorem 2.3:

In a special Kawaguchi space, if $D_s\Omega d\tau = 0$, then the relation $D\Omega = L_v\Omega d\tau$ holds.

Proof:

Setting $D_s\Omega d\tau = 0$ in equation (2.21), it follows that

(2.28) $\delta_2\Omega = \delta_3\Omega,$

Substituting this into equations (2.19) and (2.20) yields

(2.29) $D\Omega = L_v\Omega d\tau.$

Proving the result.

Remark 2.2: It should be noted that while the operators δ_1, δ_2 and δ_3 defined by equations (2.7), (2.8), (2.11), (2.12), (2.17) and (2.18) are not invariant individually, their differences, namely the covariant, Lie, and apparent differentials, are invariant.

3. Variations in a Special Kawaguchi Space:

When the operators D, $L_v d\tau$ and $D_s d\tau$ are applied to an object, the dependent objects generally undergo more intricate variations than those caused by mere dragging along or parallel displacement. These complex changes cannot be fully captured by simple transport operations. Let us study these variations more precisely.

From [9], we have the following:

Definition 3.1:

Let Ω be a tensor field defined in a neighborhood of a point $P \equiv (x^{*a})$. If the value of Ω at $x^{*a} + v^a(x)d\tau$ is denoted by $\delta_n\Omega$, and the value obtained by dragging along the field over $x^{*a} + v^a(x)d\tau$ is denoted by $\delta_1\Omega$, then the natural variation $D_n\Omega d\tau$ is defined by

(3.1) $D_n\Omega d\tau = \delta_n\Omega - \delta_1\Omega.$

Definition 3.2:

Let Ω be a tensor field defined in a neighborhood of a point $P \equiv (x^{*a})$. If the value of Ω at $x^{*a} + v^a(x)d\tau$ is denoted by $\delta_n\Omega$, and the value arising by dragging along the field over $x^{*a} + v^a(x)d\tau$ is $\delta_2\Omega$, then the absolute variation $D_a\Omega d\tau$ is defined by

(3.2) $D_a\Omega d\tau = \delta_n\Omega - \delta_2\Omega.$

Definition 3.3:

Let Ω be a tensor field defined near the point $P \equiv (x^{*a})$. If $\delta_n\Omega$ is the value of Ω at $x^{*a} + v^a(x)d\tau$ and $\delta_3\Omega$ is the value resulting from the parallel displacement from x^{*a} to $x^{*a} + v^a(x)d\tau$, then the geodesic variation $D_g\Omega d\tau$ is given by

(3.3) $D_g\Omega d\tau = \delta_n\Omega - \delta_3\Omega.$

Based on these definitions, we establish the following theorems:

Theorem 3.1:

In a special Kawaguchi space, if $D_n\Omega d\tau = 0$, then the relation $D_a\Omega d\tau = L_v\Omega d\tau$ holds.

Proof:

Substituting $D_n\Omega d\tau = 0$ into equation (3.1), we immediately obtain

(3.4) $\delta_n\Omega = \delta_1\Omega.$

Inserting (3.4) into (3.2), we have

(3.5) $D_a\Omega d\tau = \delta_1\Omega - \delta_2\Omega.$

Thus, the result follows.

(3.6) $D_a\Omega d\tau = L_v\Omega d\tau.$

Theorem 3.2:

In a special Kawaguchi space, if $D_n\Omega d\tau = 0$, then

(3.7) $D_g \Omega d\tau = D\Omega.$

Proof:

The result follows directly from equations (2.19), (3.3), and (3.4).

Theorem 3.3:

In a special Kawaguchi space, if $D_a \Omega d\tau = 0$, then the relation $D_a \Omega d\tau + L_v \Omega d\tau = 0$ holds.

Proof:

Substituting $D_a \Omega d\tau = 0$ into (3.2), we get

(3.8) $\delta_n \Omega = \delta_2 \Omega.$

Substituting (3.8) into (3.1), we find

(3.9) $D_n \Omega d\tau + (\delta_1 \Omega - \delta_2 \Omega) = 0.$

From equations (2.20) and (3.9), it follows that

(3.10) $D_n \Omega d\tau + L_v \Omega d\tau = 0.$

Theorem 3.4:

In a special Kawaguchi space, if $D_a \Omega d\tau = 0$, then

(3.11) $D_g \Omega d\tau + D_s \Omega d\tau = 0$ holds.

Proof:

The result follows immediately from equations (2.21), (3.3), and (3.8).

Theorem 3.5:

In a special Kawaguchi space, if $D_g \Omega d\tau = 0$, then the relation $D_n \Omega d\tau + D\Omega = 0$ holds.

Proof:

Substituting $D_g \Omega d\tau = 0$ into (3.3), we get

(3.12) $\delta_n \Omega = \delta_3 \Omega,$

Inserting (3.12) into (3.1), we obtain

(3.13) $D_n \Omega d\tau + (\delta_1 \Omega - \delta_3 \Omega) = 0.$

From equations (2.19) and (3.13), we deduce

(3.14) $D_n \Omega d\tau + D\Omega = 0.$

Theorem 3.6:

In a special Kawaguchi space, if the geodesic variation vanishes then the following relation is satisfied

(3.15) $D_a \Omega d\tau - D_s \Omega d\tau = 0.$

Proof:

The result follows directly from equations (2.21), (3.2), and (3.12).

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